1
A One-Dimensional Optical Cavity with Output Coupling: Classical Analysis

In this chapter, a one-dimensional optical cavity with output coupling is considered. The optical cavity has transmission loss at one or both of the end surfaces. The classical, natural cavity mode is defined, and decaying or growing mode functions are derived using the cavity boundary conditions. A series of resonant modes appears. But these modes are not orthogonal to each other and are not suitable for quantum-mechanical analysis of the optical field inside or outside of the cavity. Hypothetical boundaries are added at infinity in order to obtain orthogonal wave mode functions that satisfy the cavity and infinity boundary conditions. These new mode functions are suitable for field quantization, where each mode is quantized separately and the electric field of an optical wave is made up of contributions from each mode. Some results of quantization are described in the next chapter. Chapter 3 deals with the usual quasimode model: a perfect cavity with distributed internal loss or with a fictitious loss reservoir.

1.1
Boundary Conditions at Perfect Conductor and Dielectric Surfaces

In a source-free space, the electric field **E** and the magnetic field **H** described using a vector potential **A** satisfy the following equations:

$$\nabla^2 \mathbf{A}(\mathbf{r}) - \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 \mathbf{A}(\mathbf{r}) = 0 \tag{1.1}$$

$$\mathbf{E}(\mathbf{r}) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}) \tag{1.2}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu}\nabla \times \mathbf{A}(\mathbf{r}) \tag{1.3}$$

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where c is the velocity of light and μ is the magnetic permeability of the medium. We work in a Coulomb gauge where

$$\operatorname{div}\mathbf{A}(\mathbf{r}) = 0 \tag{1.4}$$

In this chapter we consider one-dimensional, plane vector waves that are polarized in the x-direction and propagated to the z-direction. Therefore we write

$$\mathbf{A}(\mathbf{r}) = A(z,t)\mathbf{x} \tag{1.5}$$

where $\mathbf{x}$ is the unit vector in the x-direction. At the surface of a perfect conductor that is vertical to the z-axis, the tangential component of the electric vector vanishes. The tangential component of the magnetic field should be proportional to the surface current. In the absence of a forced current, this condition is automatically satisfied: the magnetic field that is consistent with the electric field induces the necessary surface current. At the interface between two dielectric media, or at the interface between a dielectric medium and vacuum, the tangential components of both the electric and magnetic fields must be continuous. Thus, at the surface z_c of a perfect conductor,

$$\frac{\partial}{\partial t}A(z_c,t) = 0 \tag{1.6}$$

and at the interface z_i of dielectrics 1 and 2,

$$\frac{\partial}{\partial t}A_1(z_i,t) = \frac{\partial}{\partial t}A_2(z_i,t) \tag{1.7}$$

$$\left.\frac{\partial}{\partial z}A_1(z,t)\right|_{z=z_i} = \left.\frac{\partial}{\partial z}A_2(z,t)\right|_{z=z_i} \tag{1.8}$$

In Equation 1.8 we have dropped the magnetic permeability μ_1 and μ_2, assuming that both of them are equal to that in vacuum, μ_0, which is usually valid in the optical region of the frequency spectrum.

1.2 Classical Cavity Analysis

1.2.1 One-Sided Cavity

Consider a one-sided cavity depicted in Figure 1.1. This cavity consists of a lossless non-dispersive dielectric of dielectric constant ε_1, which is bounded by a perfect conductor at $z = -d$ and vacuum at $z = 0$. The outer space $0 < z$ is a vacuum of dielectric constant ε_0. Subscripts 1 and 0 will be used for the regions $-d < z < 0$ and $0 < z$, respectively. The velocity of light in the regions 1 and 0 are c_1 and c_0, respectively.

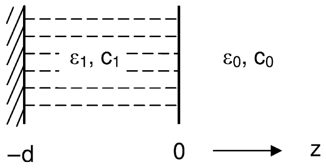

Figure 1.1 The one-sided cavity model.

The natural oscillating field mode of the cavity, the cavity resonant mode, is defined as the mode that has only an outgoing wave in the outer space $0 < z$. For reasons that will be described in Chapter 14, we also derive a mode that has only an incoming wave outside. For simplicity, let us call these the outgoing mode and incoming modes, respectively. Let the mode functions be

$$\begin{aligned} A(z,t) &= u(z)e^{-i\omega t}, \qquad -d < z < 0 \\ &= ve^{-i(\omega t \mp k_0 z)}, \qquad 0 < z \end{aligned} \tag{1.9}$$

where v is a constant. We define the wavenumber k by

$$k_i = \omega/c_i, \qquad i = 0, 1 \tag{1.10}$$

The upper and lower signs in the second line in Equation 1.9 are for the outgoing mode and the incoming mode, respectively. Substituting Equation 1.9 into Equation 1.1 via Equation 1.5 we obtain

$$\begin{aligned} -\frac{\omega^2}{c_1^2}u &= \left(\frac{d}{dz}\right)^2 u, \qquad -d < z < 0 \\ k_0 &= \frac{\omega}{c_0}, \qquad 0 < z \end{aligned} \tag{1.11}$$

Thus we can set

$$\begin{aligned} u(z) &= Ae^{ik_1 z} + Be^{-ik_1 z} \\ v &= C \end{aligned} \tag{1.12}$$

where $k_1 = \omega_k / c_1$. Putting this into Equation 1.6 for $z = -d$ and into Equations 1.7 and 1.8 for $z = 0$, we obtain

$$\begin{aligned} Ae^{-ik_1 d} + Be^{ik_1 d} &= 0 \\ A + B &= C \\ ik_1(A - B) &= \pm ik_0 C \end{aligned} \tag{1.13}$$

We then have

$$e^{2ik_1 d} = \frac{\mp k_0 - k_1}{k_1 \mp k_0} = \frac{\mp c_1 - c_0}{c_0 \mp c_1} \tag{1.14}$$

For the outgoing mode (upper sign) we have

$$e^{2ik_1 d} = \frac{-c_1 - c_0}{c_0 - c_1} = -\frac{c_0 + c_1}{c_0 - c_1} \tag{1.15}$$

Because we are assuming that both c_1 and c_0 are real and that the velocity of light in the dielectric is smaller than that in vacuum ($c_1 < c_0$), k_1 is a complex number K_{1out}. We reserve k_1 for the real part of K_{1out}. Then we obtain

$$\begin{aligned} K_{1out,m} &= k_{1m} - i\gamma \\ k_{1m} &= \frac{1}{2d}(2m+1)\pi, \qquad m = 0, 1, 2, 3, \ldots \\ \gamma &= \frac{1}{2d}\ln\left(\frac{c_0 + c_1}{c_0 - c_1}\right) = \frac{1}{2d}\ln\left(\frac{1}{r}\right) \end{aligned} \tag{1.16}$$

There is an eigenmode every π/d in the wavenumber. Note that the imaginary part is independent of the mode number. The coefficient

$$r = \frac{c_0 - c_1}{c_0 + c_1} \tag{1.17}$$

is the amplitude reflectivity of the coupling surface, $z = 0$, for the wave incident from the left, that is, from inside the cavity. The corresponding eigenfrequency of the mode is

$$\begin{aligned} \Omega_m \equiv \Omega_{kout,m} &= \omega_{cm} - i\gamma_c \\ \omega_{cm} &= \frac{c_1}{2d}(2m+1)\pi, \qquad m = 0, 1, 2, 3, \ldots \\ \gamma_c &= \frac{c_1}{2d}\ln\left(\frac{c_0 + c_1}{c_0 - c_1}\right) = \frac{c_1}{2d}\ln\left(\frac{1}{r}\right) \end{aligned} \tag{1.18a}$$

where we have defined the complex angular frequency Ω_m. In subsequent chapters, a typical cavity eigenfrequency, with a certain large number m, will be denoted as

$$\Omega_c = \omega_c - i\gamma_c \tag{1.18b}$$

The separation of the mode frequencies is $\Delta\omega_c = c_1\pi/d$.

Likewise, for the incoming mode (lower sign) we have

$$e^{2ik_1 d} = \frac{c_1 - c_0}{c_0 + c_1} = -\frac{c_0 - c_1}{c_0 + c_1} \tag{1.19}$$

from which we obtain

$$K_{1in,m} = K^*_{1out,m} = k_{1m} + i\gamma \tag{1.20a}$$

and

$$\Omega_{kin,m} = \Omega^*_{kout,m} = \omega_{cm} + i\gamma_c \equiv \Omega^*_m \tag{1.20b}$$

Going back to Equation 1.13 we now get the ratios of A, B, and C. Thus, except for an undetermined constant factor, for the outgoing mode we have

$$A(z,t) = u_m(z)e^{-i\Omega_m t} \tag{1.21a}$$

$$u_m(z) = \begin{cases} \sin\{\Omega_m(z+d)/c_1\}, & -d < z < 0 \\ \sin\{\Omega_m d/c_1\}e^{i\Omega_m(z/c_0)}, & 0 < z \end{cases} \tag{1.21b}$$

and for the incoming mode we have

$$A(z,t) = \tilde{u}_m(z)e^{-i\Omega_m^* t} \tag{1.22a}$$

$$\tilde{u}_m(z) = \begin{cases} \sin\{\Omega_m^*(z+d)/c_1\}, & -d < z < 0 \\ \sin\{\Omega_m^* d/c_1\}e^{-i\Omega_m^*(z/c_0)}, & 0 < z \end{cases} \tag{1.22b}$$

where the suffix m signifies the cavity mode. We note that the outgoing mode is temporally decaying whereas the incoming mode is growing. Inside the cavity, the field is a superposition of a pair of right-going and left-going waves with decaying or growing amplitudes. We note that $\tilde{u}_m(z) = u_m^*(z)$, meaning that the complex conjugate of the incoming mode function is the time-reversed outgoing mode function.

We also note that different members of the outgoing mode are non-orthogonal in the sense that

$$\int_{-d}^{0} u_m^*(z)\, u_{m'}(z)dz \neq 0, \qquad m \neq m' \tag{1.23}$$

Similarly, members of the incoming mode are mutually non-orthogonal. However, a member of the outgoing mode and a member of the incoming mode are approximately orthogonal. That is, if normalized properly, it can be shown that

$$\int_{-d}^{0} \tilde{u}_{out,m}^*(z)\, u_{in,m'}(z)dz \cong \delta_{m,m'} \tag{1.24}$$

The approximation here neglects the integrals of spatially rapidly oscillating terms. This is justified when the cavity length d is much larger than the optical wavelength $\lambda_k = 2\pi c_1/\omega_k$ or when $m \gg 1$ in Equation 1.16. These relationships among the outgoing and incoming mode functions will be discussed in Chapter 14 in relation to the quantum excess noise or the excess noise factor of a laser.

1.2.2
Symmetric Two-Sided Cavity

Consider a symmetrical, two-sided cavity depicted in Figure 1.2. This cavity consists of a lossless non-dispersive dielectric of dielectric constant ε_1, which is

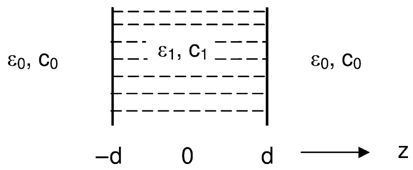

Figure 1.2 The symmetrical two-sided cavity model.

bounded by external vacuum at both $z = -d$ and $z = d$. Subscripts 1 and 0 will be used for the internal region $-d < z < d$ and external region $d < z$ and $z < -d$, respectively. The velocity of light in the regions 1 and 0 are c_1 and c_0, respectively.

Let the mode functions be

$$\begin{aligned} A(z,t) &= u(z)e^{-i\omega t}, & -d < z < d \\ &= ve^{-i(\omega t \mp k_0 z)}, & d < z \\ &= we^{-i(\omega t \pm k_0 z)}, & z < -d \end{aligned} \tag{1.25}$$

where again the upper signs are for the outgoing mode and the lower ones are for the incoming mode, and both v and w are constants. Following a similar procedure as above, this time we get symmetric and antisymmetric mode functions for both outgoing and incoming modes.

The symmetric outgoing mode function is (problem 1-1)

$$A(z,t) = \begin{cases} \cos(\Omega z/c_1)e^{-i\Omega t}, & -d < z < d \\ \cos(\Omega d/c_1)e^{-i\Omega\{t-(z-d)/c_0\}}, & d < z \\ \cos(\Omega d/c_1)e^{-i\Omega\{t+(z+d)/c_0\}}, & z < -d \end{cases} \tag{1.26}$$

where

$$\begin{aligned} \Omega &= \Omega_m = \omega_m - i\gamma_c \\ \omega_m &= \frac{c_1}{d} m\pi, \qquad m = 0, 1, 2, 3, \ldots \\ \gamma_c &= \frac{c_1}{2d}\ln\left(\frac{c_0 + c_1}{c_0 - c_1}\right) = \frac{c_1}{2d}\ln\left(\frac{1}{r}\right) \end{aligned} \tag{1.27}$$

The antisymmetric outgoing mode function is

$$A(z,t) = \begin{cases} \sin(\Omega z/c_1)e^{-i\Omega t}, & -d < z < d \\ \sin(\Omega d/c_1)e^{-i\Omega\{t-(z-d)/c_0\}}, & d < z \\ -\sin(\Omega d/c_1)e^{-i\Omega\{t+(z+d)/c_0\}}, & z < -d \end{cases} \tag{1.28}$$

where

$$\begin{aligned}\Omega &= \Omega_m = \omega_m - i\gamma_c \\ \omega_m &= \frac{c_1}{2d}(2m+1)\pi, \qquad m = 0, 1, 2, 3, \ldots \\ \gamma_c &= \frac{c_1}{2d}\ln\left(\frac{c_0+c_1}{c_0-c_1}\right) = \frac{c_1}{2d}\ln\left(\frac{1}{r}\right)\end{aligned} \tag{1.29}$$

The symmetric and antisymmetric incoming mode functions are given by Equations 1.26 and 1.28, respectively, with Ω_m replaced by Ω_m^*. Note that the antisymmetric mode functions for $0 < z$, if shifted to the left by d ($z \to z+d$), coincide with the mode functions for the one-sided cavity in Equations 1.21a and 1.22a, as is expected from the mirror symmetry of the two-sided cavity. The relations 1.23 and 1.24 also hold in this cavity model.

1.3 Normal Mode Analysis: Orthogonal Modes

As we have seen in the previous section, the natural resonant modes (outgoing mode) of the cavity, as well as the associated incoming modes, are non-orthogonal and associated with time-decaying or growing factors. This feature is not suitable for straightforward quantization. For straightforward quantization, we need orthogonal, stationary modes describing the cavity. For this purpose, we introduce artificial boundaries at large distances so as to get such field modes.

1.3.1 One-Sided Cavity

1.3.1.1 Mode Functions of the "Universe"

For the one-sided cavity, we add a perfectly reflective boundary of a perfect conductor at $z = L$ as in Figure 1.3. Then we have three boundaries: at $z = -d$ and $z = L$ the boundary condition 1.6 applies, whereas at $z = 0$ the conditions 1.7 and 1.8 apply. The region $-d < z < L$ is our "universe," within which the region $-d < z < 0$ is the cavity and the region $0 < z < L$ is the outside space.

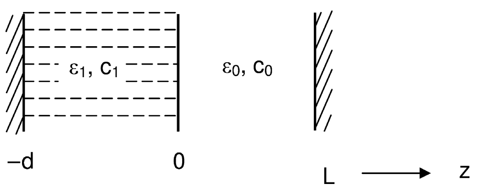

Figure 1.3 The one-sided cavity embedded in a large cavity.

Here, again, subscripts 1 and 0 will be used for the regions $-d < z < 0$ and $0 < z < L$, respectively. Assuming, again, the form of Equation 1.5 for the field, we assume the following form of the field:

$$A(z,t) = Q(t)U(z) \tag{1.30}$$

We try solutions of the form:

$$A_1(z,t) = Q(t)U_1(z,t), \quad -d < z < 0 \tag{1.31a}$$

$$A_0(z,t) = Q(t)U_0(z,t), \quad 0 < z < L \tag{1.31b}$$

Then Equation 1.1 gives

$$\left(\frac{d}{dt}\right)^2 Q(t) + \omega^2 Q(t) = 0 \tag{1.32a}$$

and

$$\begin{aligned} &\left(\frac{d}{dz}\right)^2 U_1(z) + (k_1)^2 U_1(z) = 0 \\ &\left(\frac{d}{dz}\right)^2 U_0(z) + (k_0)^2 U_0(z) = 0 \end{aligned} \tag{1.32b}$$

where

$$k_i = \omega/c_i = \omega(\varepsilon_i \mu_0)^{1/2}, \quad i = 0, 1 \tag{1.33}$$

Thus we assume the following spatial form:

$$U(z) = \begin{cases} U_1(z) = a_1 e^{ik_1 z} + b_1 e^{-ik_1 z}, & -d < z < 0 \\ U_0(z) = a_0 e^{ik_0 z} + b_0 e^{-ik_0 z}, & 0 < z < L \end{cases} \tag{1.34}$$

Applying the boundary conditions yields

$$a_1 e^{-ik_1 d} + b_1 e^{ik_1 d} = 0 \tag{1.35a}$$

$$a_1 + b_1 = a_0 + b_0 \tag{1.35b}$$

$$a_1 k_1 - b_1 k_1 = a_0 k_0 - b_0 k_0 \tag{1.35c}$$

$$a_0 e^{ik_0 L} + b_0 e^{-ik_0 L} = 0 \tag{1.35d}$$

For non-vanishing coefficients, we need the determinantal equation (problem 1-2)

$$\tan(k_0 L) = -(k_0/k_1)\tan(k_1 d) \tag{1.36}$$

or

$$c_1 \tan \frac{\omega d}{c_1} + c_0 \tan \frac{\omega L}{c_0} = 0 \tag{1.37}$$

Under this condition, the function A can be determined except for a constant factor as

$$\begin{aligned} A_1(z,t) &= f \sin k_1(z+d)\cos(\omega t+\phi), && -d < z < 0 \\ A_0(z,t) &= f\frac{k_1 \cos k_1 d}{k_0 \cos k_0 L}\sin k_0(z-L)\cos(\omega t+\phi) \\ &= f\left(\frac{k_1}{k_0}\cos k_1 d \sin k_0 z + \sin k_1 d \cos k_0 z\right)\cos(\omega t+\phi), && 0 < z < L \end{aligned} \tag{1.38}$$

where ϕ is an arbitrary phase and f is an arbitrary constant. Equation 1.37 has been used in the last line.

1.3.1.2 Orthogonal Spatial Modes of the "Universe"

Now the allowed values of $k_{0,1}$ or ω are determined by Equation 1.37. If we choose a large L, $L \gg d$, it can be seen that the solution is distributed rather uniformly with approximate frequency, in k_0, of π/L, and that there is no degeneracy in k_0 and thus in ω. It can be shown that the space part of the jth mode functions in Equation 1.38, that is,

$$U_j(z) = \begin{cases} \sin k_{1j}(z+d), & -d < z < 0 \\ \left(\frac{k_{1j}}{k_{0j}}\cos k_{1j} d \sin k_{0j} z + \sin k_{1j} d \cos k_{0j} z\right), & 0 < z < L \end{cases} \tag{1.39}$$

form an orthogonal set in the sense that

$$\int_{-d}^{L} \varepsilon(z) U_i(z) U_j(z) dz = 0, \qquad i \neq j \tag{1.40a}$$

To show this relation, let us consider the integral

$$\begin{aligned} I &= \int_{-d}^{L} \frac{1}{\mu_0}\frac{\partial}{\partial z} U_i(z) \frac{\partial}{\partial z} U_j(z) dz \\ &= \frac{1}{\mu_0} U_i(z) \frac{\partial}{\partial z} U_j(z)\Big|_{-d}^{0} + \frac{1}{\mu_0} U_i(z) \frac{\partial}{\partial z} U_j(z)\Big|_{0}^{L} \\ &\quad - \frac{1}{\mu_0}\int_{-d}^{0} U_i(z)\left(\frac{\partial}{\partial z}\right)^2 U_j(z) dz - \frac{1}{\mu_0}\int_{0}^{L} U_i(z)\left(\frac{\partial}{\partial z}\right)^2 U_j(z) dz \end{aligned}$$

$$= \frac{k_{1j}^2}{\mu_0}\int_{-d}^{0} U_i(z)\,U_j(z)dz \;+\; \frac{k_{0j}^2}{\mu_0}\int_{0}^{L} U_i(z)\,U_j(z)dz$$

$$= \omega_j^2\left(\int_{-d}^{0} \varepsilon_1\,U_i(z)\,U_j(z)dz \;+\int_{0}^{L} \varepsilon_0\,U_i(z)\,U_j(z)dz\right) \tag{1.40b}$$

$$= \omega_j^2\int_{-d}^{L} \varepsilon(z)\,U_i(z)\,U_j(z)dz$$

In the second line, the values at $z=-d$ and $z=L$ vanish because of the condition on the perfect boundary, while the values at $z=0$ cancel because of the continuity of both the function and its derivative. The Helmholtz equation 1.32a and 1.32b was used on going from the third to the fourth line. Finally, Equation 1.33 was used to go to the fifth line. Because we can interchange $U_i(z)$ and $U_j(z)$ in the first line, we also have

$$I = \omega_i^2\int_{-d}^{L} \varepsilon(z)\,U_i(z)\,U_j(z)dz \tag{1.40c}$$

Thus we have

$$0 = \left(\omega_j^2 - \omega_i^2\right)\int_{-d}^{L} \varepsilon(z)\,U_i(z)\,U_j(z)dz \tag{1.40d}$$

Since the modes are non-degenerate, the integral must vanish, which proves Equation 1.40a.

1.3.1.3 Normalization of the Mode Functions of the "Universe"

For later convenience, we normalize the mode function 1.39 as

$$U_j(z) = N_j u_j(z) \tag{1.41a}$$

$$u_j(z) = \begin{cases} \sin k_{1j}(z+d), & -d<z<0 \\ \left(\dfrac{k_{1j}}{k_{0j}}\cos k_{1j}d\,\sin k_{0j}z + \sin k_{1j}d\,\cos k_{0j}z\right), & 0<z<L \end{cases} \tag{1.41b}$$

with the orthonormality property

$$\int_{-d}^{L} \varepsilon(z)\,U_i(z)\,U_j(z)dz = \delta_{i,j} \tag{1.42a}$$

where the Kronecker delta symbol

$$\delta_{i,j} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \tag{1.42b}$$

It will be left for the reader to derive the normalization constant (problem 1-3):

$$
\begin{aligned}
N_j &= \sqrt{\frac{2}{\varepsilon_1\{d + (\cos k_{1j}d/\cos k_{0j}L)^2 L\}}} \\
&= \sqrt{\frac{2}{\varepsilon_1\{d + (1 - K\sin^2 k_{1j}d)L\}}} \qquad (1.43) \\
K &= 1 - \left(\frac{k_{0j}}{k_{1j}}\right)^2 = 1 - \left(\frac{c_1}{c_0}\right)^2
\end{aligned}
$$

The condition 1.37 has been used in the second line. Note that K is a constant for a given cavity. As will be discussed in Section 1.4, we will take the limit $L \to \infty$ and ignore the quantity d in Equation 1.43 in later applications of the one-sided cavity model.

1.3.1.4 Expansion of the Field in Terms of Orthonormal Mode Functions and the Field Hamiltonian

If the mode functions in Equation 1.41a form a complete set, which will be discussed in the last part of this section, a vector potential of any spatio-temporal distribution in the entire space $-d \le z \le L$, which vanishes at both ends, may be expanded in terms of these functions in the form

$$A(z,t) = \sum_k Q_k(t)\, U_k(z) \qquad (1.44)$$

where $Q_k(t)$ is the time-varying expansion coefficient. The corresponding electric and magnetic fields are found from Equations 1.2 and 1.3. In the following, we want to calculate the total Hamiltonian associated with the waves in Equations 1.39:

$$
\begin{aligned}
H &= \int_{-d}^{L} \left[\frac{\varepsilon}{2} E(z,t)^2 + \frac{\mu}{2} H(z,t)^2\right] dz \\
&= \int_{-d}^{L} \left[\frac{\varepsilon}{2}\left(\frac{\partial}{\partial t} A(z,t)\right)^2 + \frac{1}{2\mu}\left(\frac{\partial}{\partial z} A(z,t)\right)^2\right] dz
\end{aligned} \qquad (1.45)
$$

Writing

$$\frac{d}{dt} Q_k = P_k \qquad (1.46)$$

we perform the integrations in Equation 1.45, which include, for the regions both inside and outside the cavity, the squared electric and magnetic fields for every member k and cross-terms of electric fields coming from different members k and k', and similar cross-terms for the magnetic field. The integration is done in

Appendix A. The resultant expression is very simple due to the orthogonality of the mode functions:

$$H = \frac{1}{2}\sum_k \left(P_k^2 + \omega_k^2 Q_k^2\right) \tag{1.47}$$

1.3.2
Symmetric Two-Sided Cavity

1.3.2.1 Mode Functions of the "Universe"

For the symmetric two-sided cavity, we impose a periodic boundary condition instead of perfect boundary conditions. Figure 1.4 depicts a two-sided cavity of a lossless non-dispersive dielectric of dielectric constant ε_1 extending from $z = -d$ to $z = d$. The exterior regions are vacuum with dielectric constant ε_0. We assume a periodicity with period $L + 2d$ and set another dielectric from $z = L + d$ to $z = L + 3d$. The region $-d < z < L + d$ is one period of our "universe" within which the region $-d < z < d$ is the cavity. The "universe" may alternatively be thought to exist in the symmetric region $-L/2 - d < z < L/2 + d$.

Here, again, subscripts 1 and 0 will be used for the regions $-d < z < d$ and $d < z < L + d$, respectively. Assuming again the form of Equation 1.5 for the field, we assume a solution of the form

$$A(z,t) = Q_k(t)\,U_k(z) \tag{1.48}$$

Equation 1.1 then yields

$$\left(\frac{d}{dt}\right)^2 Q_j(t) = -\omega_j^2 Q_j(t) \tag{1.49}$$

$$\left(\frac{d}{dz}\right)^2 U_j(z) = -k_j^2 U_j(z) \tag{1.50}$$

where $k_j = \omega_j/c$. A general solution of Equation 1.50 in the one period may be written as

$$U_{0j}(z) = A_j e^{ik_{0j}z} + B_j e^{-ik_{0j}z} \quad (d < z < L + d) \tag{1.51}$$

$$U_{1j}(z) = C_j e^{ik_{1j}z} + D_j e^{-ik_{1j}z} \quad (-d < z < d) \tag{1.52}$$

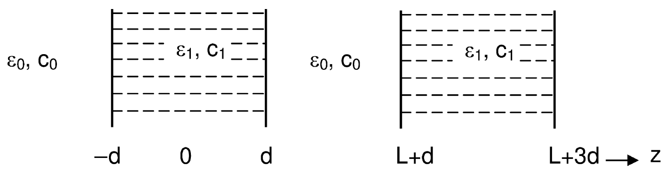

Figure 1.4 The two-sided cavity with the cyclic boundary condition.

where

$$k_{0,1j} = \omega_j / c_{0,1} \tag{1.53}$$

Applying the continuity boundary conditions at $z = d$ and the periodic boundary conditions at $z = -d$ and $Z = L + d$, one has

$$\begin{aligned} U_{1j}(d) &= U_{0j}(d) \\ U'_{1j}(d) &= U'_{0j}(d) \\ U_{1j}(-d) &= U_{0j}(L+d) \\ U'_{1j}(-d) &= U'_{0j}(L+d) \end{aligned} \tag{1.54}$$

The last two equations are obtained by combining the continuous conditions at $z = -d$ with the cyclic boundary conditions. With Equations 1.51 and 1.52, the coefficients A_j, B_j, C_j, and D_j must satisfy

$$\begin{aligned} C_j e^{ik_{1j}d} + D_j e^{-ik_{1j}d} &= A_j e^{ik_{0j}d} + B_j e^{-ik_{0j}d} \\ C_j k_{1j} e^{ik_{1j}d} - D_j k_{1j} e^{-ik_{1j}d} &= A_j k_{0j} e^{ik_{0j}d} - B_j k_{0j} e^{-ik_{0j}d} \\ C_j e^{-ik_{1j}d} + D_j e^{ik_{1j}d} &= A_j e^{ik_{0j}(L+d)} + B_j e^{-ik_{0j}(L+d)} \\ C_j k_{1j} e^{-ik_{1j}d} - D_j k_{1j} e^{ik_{1j}d} &= A_j k_{0j} e^{ik_{0j}(L+d)} - B_j k_{0j} e^{-ik_{0j}(L+d)} \end{aligned} \tag{1.55}$$

It is left to the reader to show that the determinantal equation for non-zero values of the coefficients is

$$\left(1 - \frac{k_{1j}}{k_{0j}}\right)^2 \sin^2\left(k_{1j}d - \frac{k_{0j}L}{2}\right) = \left(1 + \frac{k_{1j}}{k_{0j}}\right)^2 \sin^2\left(k_{1j}d + \frac{k_{0j}L}{2}\right) \tag{1.56}$$

which reduces to two equations:

$$\tan(k_{1j}d) = -\frac{c_0}{c_1}\tan\left(\frac{k_{0j}L}{2}\right) \qquad (a \text{ mode}) \tag{1.57a}$$

$$\tan(k_{1j}d) = -\frac{c_1}{c_0}\tan\left(\frac{k_{0j}L}{2}\right) \qquad (b \text{ mode}) \tag{1.57b}$$

Thus we have two sets of eigenvalues of wavenumber k_j or eigenfrequency ω_j. We refer to the modes determined by Equation 1.57a as *a* modes and those determined by Equation 1.57b as *b* modes. Graphical examination shows that the *a* mode and *b* mode solutions appear alternately on the angular frequency axis. Then we derive mode functions from Equations 1.55 and 1.57a and 1.57b as (problem 1-4):

$$U_j^a(z) = \alpha_j \times \begin{cases} \sin(k_{1j}z) & (-d < z < d) \\ \sin(k_{1j}d)\cos\{k_{0j}(z-d)\} & \\ +\dfrac{c_0}{c_1}\cos(k_{1j}d)\sin\{k_{0j}(z-d)\} & (d < z < L + d) \end{cases} \tag{1.58a}$$

$$U_j^b(z) = \beta_j \times \begin{cases} \cos(k_{1j}z) & (-d < z < d) \\ \cos(k_{1j}l)\cos\{k_{0j}(z-d)\} & \\ -\dfrac{c_0}{c_1}\sin(k_{1j}l)\sin\{k_{0j}(z-d)\} & (d < z < L + d) \end{cases} \tag{1.58b}$$

1.3.2.2 Orthonormal Spatial Modes of the "Universe"

It can be shown that the two different members, each from either a mode or b mode, are orthogonal in the sense of Equation 1.40a. They are normalized in the sense of Equation 1.42a if the constants α_j and β_j are given by

$$\alpha_j = \sqrt{\frac{2}{\varepsilon_1\{2d + (1 - K\sin^2 k_{1j}d)L\}}} \tag{1.59a}$$

$$\beta_j = \sqrt{\frac{2}{\varepsilon_1\{2d + (1 - K\cos^2 k_{1j}d)L\}}} \tag{1.59b}$$

where K was defined in Equation 1.43. This can be derived by repeated use of the determinantal equations 1.57a and 1.57b. As will be discussed in Section 1.4, we will take the limit $L \to \infty$ and ignore the quantity $2d$ in Equations 1.59a and 1.59b in later applications of the two-sided cavity model.

1.3.2.3 Expansion of the Field in Terms of Orthonormal Mode Functions and the Field Hamiltonian

If the mode functions in Equations 1.58a and 1.58b form a complete set, which will be discussed in Section 1.6, a vector potential of any spatio-temporal distribution in the entire space $-d < z < L + d$ or $-L/2 - d < z < L/2 + d$ may be expanded in terms of these functions in the same form as in Equation 1.44,

$$A(z,t) = \sum_k Q_k(t)\,U_k(z) \tag{1.60}$$

where $Q_k(t)$ is the time-varying expansion coefficient. The total Hamiltonian defined as in Equation 1.45, with the upper limit of integration replaced by $L + d$, can be evaluated again defining the "momentum" P_k associated with the "amplitude" Q_k as in Equation 1.46. Using Equations 1.2 and 1.3, we perform the integrations as in Equation 1.45, which include, for the regions both inside and outside the cavity, the squared electric and magnetic field for every member k from both the a mode and b mode functions, and cross-terms of electric fields coming from different members k and k' and similar cross-terms for the magnetic field.

All the cross-terms vanish on integration due to the orthogonality of the mode functions. The resultant expression is the same as Equation 1.47:

$$H = \frac{1}{2}\sum_k (P_k^2 + \omega_k^2 Q_k^2) \tag{1.61}$$

Note that the mode index k here includes both a mode and b mode functions.

1.4
Discrete versus Continuous Mode Distribution

The length L, expressing the extent of the outside region, was introduced for mathematical convenience. As we have seen, this allowed us to obtain discrete, orthogonal mode functions, which are stationary. We eventually normalized them. Because the physical content of the outside region is the free space outside the cavity, there is no reason to have a finite value of L. On the contrary, if L is finite (comparable to d), various artifacts may arise due to reflections at the perfect boundary at $z = L$ in the case of one-sided cavity or at the neighboring cavity surface in the case of the two-sided cavity. For this reason, we take the limit $L \to \infty$ in what follows.

In this limit, in the case of the one-sided cavity, the normalization constant reduces to

$$N_j = \sqrt{\frac{2}{\varepsilon_1 L(1 - K\sin^2 k_{1j}d)}} \tag{1.62a}$$

and the normalized mode function is

$$U_j(z) = \sqrt{\frac{2}{\varepsilon_1 L(1 - K\sin^2 k_{1j}^2 d)}} \times \begin{cases} \sin k_{1j}(z+d), & -d < z < 0 \\ \left(\frac{k_{1j}}{k_{0j}}\cos k_{1j}d \sin k_{0j}z + \sin k_{1j}d\cos k_{0j}z\right), & 0 < z < L \end{cases} \tag{1.62b}$$

The mode distribution in the frequency domain is determined by Equation 1.37. For $L \to \infty$, the spacing $\Delta\omega$ of the two eigenfrequencies is

$$\Delta\omega = (c_0/L)\pi \tag{1.63}$$

which is infinitely small and the modes distribute continuously. The density of modes (the number of modes per unit angular frequency) is

$$\rho(\omega) = \frac{L}{\pi c_0} \tag{1.64}$$

We note that the maxima of the normalization constant N_j occur at the cavity resonant frequencies given by Equation 1.18a.

In the case of the two-sided cavity, the normalization constants become

$$\alpha_j = \sqrt{\frac{2}{\varepsilon_1 L(1 - K\sin^2 k_{1j}d)}} \tag{1.65a}$$

$$\beta_j = \sqrt{\frac{2}{\varepsilon_1 L(1 - K\cos^2 k_{1j}d)}} \tag{1.65b}$$

It is easy to see from Equations 1.57a and 1.57b that the a mode and b mode appear in pairs along the frequency axis and, in the limit $L \to \infty$, every pair is degenerate. The separation of the pairs is now

$$\Delta\omega = (2c_0/L)\pi \tag{1.66}$$

so that the density of modes is

$$\rho_a(\omega) = \rho_b(\omega) = \frac{1}{2}\rho(\omega) = \frac{L}{2\pi c_0} \tag{1.67}$$

For both the one-sided and the two-sided cavities, the overall density of modes becomes independent of the cavity size and is equal to $L/\pi c_0$.

In what follows we sometimes encounter the summation of some mode-dependent quantity B_k over modes of the "universe." Such a summation is converted to an integral as follows:

$$\sum_k B_k \to \int_0^\infty B_{\omega_k}\rho(\omega_k)d\omega_k \tag{1.68a}$$

for the case of a one-sided cavity, and

$$\begin{aligned}\sum_k B_k &\to \int_0^\infty \left\{B^a_{\omega_k}\rho_a(\omega_k) + B^b_{\omega_k}\rho_b(\omega_k)\right\}d\omega_k \\ &= \frac{1}{2}\int_0^\infty (B^a_{\omega_k} + B^b_{\omega_k})\rho(\omega_k)d\omega_k\end{aligned} \tag{1.68b}$$

for the case of a two-sided cavity. Correspondingly, the Kronecker delta symbol becomes a Dirac delta function by the rule

$$\rho(\omega_k)\delta_{k,k'} \to \delta(k - k') \tag{1.69}$$

because for a k-dependent variable f_k we should have $\sum_k f_k\delta_{k,k'} = \int dk f(k)\delta(k-k')$. We note that the maxima of the a (b) mode occur at the cavity resonant frequencies of the antisymmetric (symmetric) modes given by Equation 1.29 (Equation 1.27).

1.5
Expansions of the Normalization Factor

The squared normalization constant for the one-sided cavity, Equation 1.62a, divided by $2/(\varepsilon_1 L)$ has two expansions that are frequently used in subsequent sections and chapters (problems 1-6 and 1-7):

$$\begin{aligned}\frac{1}{1-K\sin^2 k_{1j}d} &= \frac{2c_0}{c_1}\left\{\sum_{n=0}^{\infty}\frac{1}{1+\delta_{0,n}}(-r)^n\cos 2nk_{1j}d\right\}\\ &= \sum_{m=-\infty}^{\infty}\frac{c_0\gamma_c/d}{\gamma_c^2+(\omega_j-\omega_{cm})^2}\end{aligned} \tag{1.70a}$$

where the coefficient r was defined in Equation 1.17 and $\omega_j = c_1 k_{1j}$. The coefficients ω_{cm} and γ_c were defined in Equation 1.18a. The first expansion is a Fourier series expansion and the second one in terms of cavity resonant modes comes from the Mittag–Leffler theorem [1], which states a partial fraction expansion based on the residue theory. Similar expansions exist for the normalization constants for the two-sided cavity in Equations 1.65a and 1.65b [2]. The expansion for Equation 1.65a is the same as in Equation 1.70a with $\omega_{cm} \to \omega_{cm}^a$:

$$\begin{aligned}\frac{1}{1-K\sin^2 k_{1j}d} &= \frac{2c_0}{c_1}\left\{\sum_{n=0}^{\infty}\frac{1}{1+\delta_{0,n}}(-r)^n\cos 2nk_{1j}d\right\}\\ &= \sum_{m=-\infty}^{\infty}\frac{c_0\gamma_c/d}{\gamma_c^2+(\omega_j-\omega_{cm}^a)^2}\\ \frac{1}{1-K\cos^2 k_{1j}d} &= \frac{2c_0}{c_1}\left\{\sum_{n=0}^{\infty}\frac{1}{1+\delta_{0,n}}(r)^n\cos 2nk_{1j}d\right\}\\ &= \sum_{m=-\infty}^{\infty}\frac{c_0\gamma_c/d}{\gamma_c^2+(\omega_j-\omega_{cm}^b)^2}\end{aligned} \tag{1.70b}$$

where $\omega_{cm}^a = (2m+1)(\pi c_1/2d)$ and $\omega_{cm}^b = 2m(\pi c_1/2d)$ (m is an integer); ω_{cm}^a (ω_{cm}^b) is the resonant frequency of the antisymmetric (symmetric) mode function defined in Equation 1.29 (Equation 1.27).

1.6
Completeness of the Modes of the "Universe"

Concerning the expansion of the field in terms of the mode functions of the "universe," it was mentioned above Equation 1.44 that the latter mode functions must form a complete set. Completeness of a set of functions means the possibility of expanding an arbitrary function, in a defined region of the variable(s), in terms of them. The set of orthogonal functions in Equations 1.41a and 1.41b fulfills this property. Assume that an arbitrary function $\Psi(z)$ defined in the region $-d < z < L$ is expanded as

$$\Psi(z) = \sum_i A_i U_i(z) \tag{1.71}$$

where A_i is a constant. Multiplying both sides by $\varepsilon(z) U_j(z)$ and integrating, we have

$$\int_{-d}^{L} \varepsilon(z) U_j(z) \Psi(z) dz = \int_{-d}^{L} \sum_i A_i \varepsilon(z) U_j(z) U_i(z) dz = \sum_i A_i \delta_{ji} = A_j \tag{1.72}$$

where we have used Equation 1.42a in the second equality. Substituting this result in Equation 1.71 we have

$$\begin{aligned} \Psi(z) &= \sum_i \int_{-d}^{L} \varepsilon(z') U_i(z') \Psi(z') dz' \, U_i(z) \\ &= \int_{-d}^{L} \left\{ \sum_i \varepsilon(z') U_i(z') U_i(z) \right\} \Psi(z') dz' \end{aligned} \tag{1.73}$$

Because $\Psi(z)$ is arbitrary, the quantity in the curly bracket should be a delta function:

$$\sum_i \varepsilon(z') U_i(z') U_i(z) = \delta(z' - z) \tag{1.74}$$

In integral form it reads

$$\int_0^{\infty} \varepsilon(z') U_i(z') U_i(z) \rho(\omega_i) d\omega_i = \delta(z' - z) \tag{1.75}$$

This is a necessary condition for completeness. Conversely, if Equation 1.75 holds, we can use Equation 1.73 to find the expansion coefficient in the form of Equation 1.72. Thus Equation 1.75 is also sufficient for completeness.

Whether the mode functions in Equations 1.41a and 1.41b really fulfill this condition is another problem. For example, for the case $-d < z < 0$ and $-d < z' < 0$, using Equation 1.62b, we need to show that

$$\int_0^{\infty} d\omega \frac{L}{\pi c_0} \varepsilon_1 \frac{2}{\varepsilon_1 L} \frac{1}{1 - K \sin^2 k_1 d} \sin k_1(z + d) \sin k_1(z' + d) = \delta(z' - z) \tag{1.76}$$

The squared normalization constant N_ω^2 is expanded in terms of $\cos 2nk_1 d$, $n = 0, 1, 2, 3, \ldots$, as in Equation 1.70a. So, except for constant factors, the integrand becomes a sum of integrals of the form

$$\int_0^{\infty} \cos\{2nd \pm (z - z')\} k_1 dk_1$$

or

$$\int_0^\infty \cos\{2nd \pm (z + z' + 2d)\} k_1 dk_1$$

We apply the formula [3]

$$\int_0^\infty \cos zk\, dk = \pi\delta(z) \tag{1.77}$$

Noting that $\delta(z \neq 0) = 0$, we find for the above combination of z and z' that

$$\begin{aligned} &\int_0^\infty d\omega \frac{L}{\pi c_0}\varepsilon_1 \frac{2}{\varepsilon_1 L}\frac{1}{1 - K\sin^2 k_1 d}\sin k_1(z+d)\sin k_1(z'+d) = \delta(z'-z) \\ &-d < z < 0, \quad -d < z' < 0 \end{aligned} \tag{1.78}$$

where we have discarded the term $-\delta(z + z' + 2d)$ because it is meaningful only at the perfect boundary, $z = z' = -d$, where all the fields vanish physically. Other combinations of the regions for z and z' can be examined in the same way. We have

$$\begin{aligned} \sum_k \varepsilon(z')U_k(z')U_k(z) &= \int_0^\infty \varepsilon(z')U_k(z')U_k(z)\rho(\omega_k)d\omega_k \\ &= \delta(z'-z) \end{aligned} \tag{1.79}$$

for $-d < z < L$, $-d < z' < L$, except $z = z' = 0$. The exception at $z = z' = 0$ occurs because at $z = 0$ the dielectric constant is unspecified. Also, the boundary conditions demand that the fields should be continuous across this boundary, so that a delta function at $z = 0$ is prohibited. The completeness of the mode functions in the case of two-sided cavities can similarly be examined.

▶ Exercises

1.1 For the symmetrical, two-sided cavity model, derive the resonant frequencies for the outgoing modes. Also derive the resonant frequencies of the incoming modes.

1-1. Set $u(z) = A\exp(ik_1 z) + B\exp(-ik_1 z)$. Then the boundary conditions at $z = d$ and $z = -d$ give, respectively,

$$\frac{A}{B} = \frac{k_1 \pm k_0}{k_1 \mp k_0}e^{-2ik_1 d}, \qquad \frac{A}{B} = \frac{k_1 \mp k_0}{k_1 \pm k_0}e^{2ik_1 d}$$

Therefore we have

$$e^{2ik_1 d} = +\frac{k_1 \pm k_0}{k_1 \mp k_0} \quad \text{or} \quad -\frac{k_1 \pm k_0}{k_1 \mp k_0}$$

For $e^{2ik_1 d} = +(k_1 \pm k_0)/(k_1 \mp k_0)$ we have $A = B$ and have symmetric modes. With the upper signs, a symmetric outgoing mode is obtained; and with the lower signs, an incoming symmetric mode is obtained. For $e^{2ik_1 d} = -(k_1 \pm k_0)/(k_1 \mp k_0)$ we have $A = -B$ and have antisymmetric modes. With the upper signs, an antisymmetric outgoing mode is obtained; and with the lower signs, an antisymmetric incoming mode is obtained.

1.2 Derive the determinantal equation 1.37 and the mode function in Equation 1.38.
1-2. Delete b_1 and b_0 from Equations 1.35b and 1.35c using Equations 1.35a and 1.35d and divide side by side to obtain Equation 1.36. Next express U_1 and U_0 in terms of a_1 and a_0. Determine a_0/a_1 by the modified version of Equation 1.35c to eliminate a_0. Finally, set $2ia_1 \exp(-ik_1 d) = \frac{1}{2} f \exp(-i\phi)$ to obtain Equation 1.38.

1.3 Derive the normalization constant in Equation 1.43 for the one-sided cavity model.
1-3. Use the form in the first line of Equation 1.38 for $0 < z < L$ and use the determinantal equation 1.37.

1.4 Derive the mode functions for the symmetrical two-sided cavity model given in Equations 1.58a and 1.58b.
1-4. See the solution to 1-2.

1.5 Show the orthogonality of mode functions in Equations 1.58a and 1.58b for the symmetric cavity under the cyclic boundary conditions following the example in Equations 1.40b–1.40d. In the limit $L \to \infty$, an a mode and a b mode can be degenerate. Are they orthogonal?
1-5. An a mode is antisymmetric and a b mode is symmetric with respect to the center of the cavity $z = 0$. So, if we have the symmetric region $-L/2 - d < z < L/2 + d$ as a cycle under the cyclic boundary condition, the a mode and b mode are easily seen to be orthogonal even if they are degenerate.

1.6 Show that the Fourier series expansion in Equation 1.70a for the squared normalization constant is valid.
1-6. Multiply both sides by the denominator on the left and compare the coefficients of $\cos 2nk_{1j}d$, $n = 0, 1, 2, 3, \ldots$, on both sides. Note that $K = 1 - (c_1/c_0)^2$ and $r = (c_0 - c_1)/(c_0 + c_1)$.

1.7 Show that the denominator in the squared normalization constant in Equation 1.70a vanishes at $\omega_j = \omega_{cm} \mp i\gamma_c$. That is, these ω_j are simple poles.
1-7. Rewrite the $\sin^2$ term as follows:

$$\sin^2 k_{1j}d \to \left(\frac{e^{ik_{1j}d} - e^{-ik_{1j}d}}{2i}\right)^2 = \left(\frac{e^{2i(k_{1m}-i\gamma)d} + e^{-2i(k_{1m}-i\gamma)d} - 2}{-4}\right)$$

$$e^{2i(k_{1m}-i\gamma)d} = -(1/r), \qquad e^{-2i(k_{1m}-i\gamma)d} = -r$$

Therefore

$$\sin^2 k_{1j}d = \frac{\{(1+r^2)/r\}+2}{4} = \frac{(1+r)^2}{4r} = \frac{1}{1-\{(1-r)/(1+r)\}^2} = \frac{1}{K}$$

References

1 Carrier, G.F., Krook, M., and Pearson, C.E. (**1966**) *Functions of a Complex Variable*, McGraw-Hill, New York.

2 Feng, X.P. and Ujihara, K. (**1990**) *Phys. Rev. A*, *41*, 2668–2676.

3 Heitler, W. (**1954**) *The Quantum Theory of Radiation*, 3rd edn, Clarendon, Oxford.