1
Introduction

1.1
Topics covered

The problems involving an undefined phase of a wave field arise in various applications and are well known in the literature. The so-called *phase problem* is the most common of these (see, e.g. [1–4]). It consists in the reconstruction of the phase distribution of the Fourier transform of a finite (compactly supported) function if its amplitude distribution is given (measured) on the entire real axis. This problem belongs to the classical *reconstruction* (identification) problems and requires the existence of a unique solution.

This book considers another class of inverse problems, which can be called the *optimization* (design) problems. In terms of the Fourier transform they could be formulated as problems in determining a finite complex function (or its amplitude and phase distribution only if the other is given) such that the modulus (amplitude) of its Fourier transform satisfies certain demands (e.g. it is close to a given (desired) positive (non-negative) function). As a rule such demands are formulated in the variational form, as the minimization of certain functionals. It can be seen that this formulation does not require the uniqueness of the solution. On the contrary, nonuniqueness is often desired because it provides additional degrees of freedom for making a decision.

The main applications of the considered phase optimization problems are the theory of the transmitting lines, field transformers, antennas and resonators. Of course, the theory developed in the book is not confined only to these applications.

The first publications on nonlinear inverse problems of the type considered have most likely appeared in the nineteen fifties and sixties [5–10].

In terms of mathematics, the problems are reduced to the nonlinear integral equations of the Hammerstein type [11, 12]. They have a linear kernel and nonlinear factor depending on the unknown function. As a rule, the phase of this function appears in the integrand separately. Similar equations can be found in the literature in the context of the mentioned phase problem [13, 14]. They have nonunique solutions, and the investigation of their structure and the processes of their branching are important mathematical problems [15]. A class of analytically solvable equations of this type is found and investigated in this book.

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Due to nonlinearity, the problems considered require development and the application of specific numerical methods. Besides simple iterative methods having a descriptive physical interpretation, different modifications of the Newton method (see e.g. [16]) seem to be the most useful. One of these modifications uses the singular value decomposition of the Jacobi matrix. Information about the singular value distribution allows for the detection of the branching points while solving the problem.

The theory described in the book was started from the works [48, 49] and [52, 75] concerning the two-element and multi-element phase field transformers, respectively. V. V. Semenov's ideas on transferring this approach to the antenna problems was described in his last work [53]. They were carried out in [27], where the main nonlinear integral equation for the problem of linear antenna synthesis by the desired amplitude pattern was obtained and an algorithm for its solution was proposed. The mathematical theory describing the structure of solutions to this type of equation and their branching began from [47]. During the following years the approach was intensively developed and utilized for many concrete problems. The theory and results were summarized and described in [25] and [26].

A new impulse for developing the theory of nonlinear integral equations of the type considered was provided by the works [28] and [29] where analytical solutions to one of these equations were obtained. The results were then generalized for some classes of nonlinear equations of the Hammerstein type [54] and [55].

The book is mostly based on the results obtained by the authors and their colleagues and published in journals or conference proceedings over some time. Some of these publications are referred to in the book. Several results were obtained during preparation of the book. The references to the original works, where the methods or results are described, are given where they are explained.

Chapter 2 was written by B. Z. Katsenelenbaum. Sections 3.1.3, 3.1.5.2, and 3.1.8.3 were written by Yu. P. Topolyuk. Section 3.3 was written by N. N. Voitovich. The rest of Chapter 3 was written by Yu. P. Topolyuk and N. N. Voitovich. Section 4.1 was written by O. O. Bulatsyk and N. N. Voitovich, Sections 4.2 and 4.5 were written by O. O. Bulatsyk, Yu. P. Topolyuk and N. N. Voitovich. Sections 4.3 and 4.4 were written by O. O. Bulatsyk. Section 5.1.2.1 was written by Yu. P. Topolyuk, and N. N. Voitovich. The rest of Section 5.1 was written by Yu. P. Topolyuk. M. I. Andriychuk wrote Sections 5.2.2, 5.2.3.3 and contributed to 5.2.3.2. S. A. Yaroshko wrote Section 5.4. The rest of Chapter 5 was written by N. N. Voitovich. Section 6.2.3.2 was written by B. Z. Katsenelenbaum and N. N. Voitovich, Sections 6.2.3.3 and 6.2.3.4 were written by N. N. Voitovich. The rest of Chapter 6 as well as Epilogue where written by B. Z. Katsenelenbaum. The general editing of the book was made by B. Z. Katsenelenbaum and N. N. Voitovich.

1.2
Outline of the book

The physical meaning of the problems considered in the book is described in Chapter 2. The problems are classified into three groups according to the physical objects they describe: transmission lines and field transformers, antennas, waveguides and resonators. Nontypical physical systems such as a cavity resonant antenna and a complex multielement beam transformer are also considered.

The problems are formulated as variational ones using different integral criteria, such as the mean-square difference between the desired and obtained field distributions (as a rule, amplitude only), the transmission, excitation or coupling coefficients, etc. In all the problems the phase functions (the phase distribution of the fields or the phase correction provided by the lenses or mirrors) are the optimization parameters.

For different reasons, several interesting problems are not in fact considered in this book. They include minimizing the field in certain areas of the middle zone [17] or creating zeros in certain directions of the radiation pattern [18], using the geometrical parameters of the system (including its shape) as additional optimization functions [19, 20], etc. Some of these problems are considered in the books [24–26].

The physical problems described in Chapter 2 are classified and generalized in Chapter 3 in the form of several variational problems formulated in terms of the functional operators. They are divided into three groups depending on the type of information given and restrictions imposed on the solution: (a) the problems with no restrictions on the solutions; (b) the problems with given either moduli (amplitudes) or arguments (phases) of the complex functions to be determined; (c) the 'homogeneous problems' in which no desired functions are given, but certain integral characteristics described by the eigenvalues of the problems are optimized.

The Hilbert spaces of L_2-type are used, in which the isometric and compact operator are both considered. Some questions concerning the stability of the problems are investigated. All the variational problems are reduced to the Lagrange–Euler equation describing the necessary conditions for the functional to be extremal. They are nonlinear integral or matrix equations and have nonunique solutions. The homogeneous equations for the branching points of their solutions are obtained.

Simple iterative methods for solving the problems are proposed. Most of them are relaxational, that is, the values of the functional make up a monotonic sequence on the successive approximations.

Typical particular cases of the operators are considered. The specific peculiarities of the problems are analyzed on these examples.

Chapter 4 contains results concerning the analytically solvable nonlinear equations of Hammerstein type, obtained in the previous chapter, and their generalizations. The existence of such equations is an important mathematical phenomenon, more unexpected than predicted. Note that the first equation of this type was obtained about forty years ago in [27]. However, its analytical solutions were observed only at the end of the last century [28] and [29].

The mentioned solutions are described by a polynomial of finite degree with complex nonconjugate zeros. These zeros are determined from a set of transcendental equations. The situation is similar to that for the linear equations with degenerated kernel, the solutions of which depend on a limited number of numeral parameters determined from the linear equation system. Indeed, the kernel of the equations considered here has a degenerated multiplier consisting of two addends. However, the polynomial degree defining the number of unknown parameters is bounded from above by a value connected with the number of oscillations of the functions contained in this multiplier. In the case when a solution of the considered equation is analytically continued to an entire function, then this function has only a limited number of the complex nonconjugate zeros, coinciding with zeros of the mentioned polynomial.

A general equation, for which such solutions exist, is given and its properties are investigated. Several theorems are proved. They describe the necessary and sufficient conditions for the function of the considered form to be a solution to this equation, the boundedness of the polynomial degree and the existence of the equivalent groups of solutions. Since the solutions with a polynomial of different degree exist simultaneously, the general structure of the set of solutions as well as the process of their branching can be completely investigated. The particular cases, connected with the Fourier transform and related to the above formulated physical problems, are analytically and numerically analyzed.

The theory is extended to a more complicated equation, with solutions which are determined by a polynomial of the described type and a real positive function. In this case the zeros of the polynomial together with the mentioned function are determined from a set of transcendental equations plus one integral equation. Both the transcendental and integral equations depend linearly on this unknown function. Numerical results for a concrete problem are presented.

The numerical methods and algorithms intended for solving the nonlinear problems of the considered types are described in Chapter 5. In the theoretical part of the chapter, two methods, namely, the steepest descent method for the direct minimization of the functionals and the simple iteration method for the solution of nonlinear equations, are investigated. In both cases the problems are complicated by the nonuniqueness of the solutions. The theorems about the convergence of the method and the accuracy of its estimations are proved. It is shown that the convergence rate decreases infinitely when approaching the branching points.

The rest of the chapter concerns the numerical solutions of the concrete problems formulated in Chapter 2. Of course, not all these problems can be solved in the context of one book. However, the authors try to overview all the key problems and show the methods which can be applied to their solution.

The simplest methods which can be applied to almost all of the problems considered in the book are the iterative methods described in the previous chapters. They are simply realizable, have clear physical interpretation and, owing to their monotonicity, almost always converge to an extremum (at least, to the local one). However, their disadvantages are the mentioned slow convergence near the branching points and nonapplicability when finding other solutions of the nonlinear equations, except those describing the extrema of the functional generating these equations.

The most powerful technique is the modified Newton method based on the singular value decomposition, permitting its application also near the branching points, by passing through them along a solution branch with almost no loss of the convergence rate. As the numerical experiments show, these points are indicated with a satisfactory accuracy. The disadvantages of this method are its sensitivity to the initial approximations and more costly result.

Combining both the above methods allows us to solve the nonlinear integral equations obtained in almost all problems considered. In this book, the combination was applied to problems of the amplitude-phase synthesis of the circular antenna, the two-elements phase field transformers and phased antenna arrays.

The most complicated problems concerning multi-element field (beam) transformers are solved by the modified iterative method in combination with the method of local variations, called together as the method of opposite directions. When optimizing an element of the system, this method compactly (in volume and time) uses and transforms the information about the other two parts, located on both sides of the optimized element.

An original method of the generalized separation of variables, intended for solving, multi-dimensional linear and nonlinear problems is described and applied to the amplitude-phase optimization of the two-dimensional antenna arrays. The method expresses the solution as a sum of functions of separated variables, which are determined successively from a one-dimensional system of nonlinear equations. The number of equations equals the dimensionality of the initial problem.

Two 'homogeneous' problems concerning the optimal shapes of the lenses in the waveguide and open resonator are solved and analyzed.

The main peculiarities of the methods and properties of the solution are mostly illustrated by two-dimensional problems. For some problems, the reader is referred to the literature where they are solved.

Finally, Chapter 6 somewhat extends the topic given by the title of the book. It considers nonstandard inverse problems formulated with respect to other (different from the phase) physical characteristics of the objects considered. They can be constructive parameters, such as impedance, orientation or the transparency of the gratings, etc.

Greatest attention is paid to the problem about the minimization of the back scattering from bodies with irregular complex impedance. A new nonlinear integral equation system is given for the impedance distribution providing the zero back scattering for all orientations of the body with respect to the source. The method is illustrated by examples of circular and elliptic cylinders.

Problems in matching the variable impedance with the incident field and applying the short-periodical grids and arrays for transformation of the linear polarization into a circular one are also considered. The chapter is unconnected with previous ones and can be read independently.

In Epilogue an attempt is made to systemize different types of the interpersonal relations in the scientific collectives. Some painful questions related to coauthorship, supervision, scientific discussions and consultations etc., are considered. These questions have the same important meaning for the scientists, as the concrete mathematical and physical problems considered in the main part of the book.