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Critical Behaviour of Conductivity and Dielectric Constant near the Metal–Non-Metal Transition Threshold

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A system consisting of randomly distributed metallic and dielectric regions is considered. The metal–non-metal transition takes place when the volume fraction of the metallic phase approaches the percolation threshold. It is shown that the static dielectric constant diverges near the threshold. Critical indexes are introduced which describe the behaviour of the conductivity and the dielectric constant near the threshold as functions of the volume fraction and frequency. The case of non-zero dc conductivity of dielectric regions is considered also. It is shown that all indexes describing the critical behaviour of complex conductivity can be expressed by two indexes which are known from computer and model experiments. The results of computer calculations of Webman et al. are analysed.

Рассмотрен случай, когда переход металл–диэлектрик происходит за счет того, что вследствие увеличения доли объема, занятого случайно расположенными в диэлектрической среде металлическими участками, по этим участкам возникает протекание. Показано, что статическая диэлектрическая проницаемость обращается в бесконечность в точке перехода. Найдены критические индексы, описывающие поведение электропроводности и диэлектрической проницаемости при подходе к точке перехода и в самой точке перехода при частоте, стремящейся к нулю. Изучен также случай, когда статическая электропроводность неметаллических участков отлична от нуля. Показано, что вся совокупность индексов, описывающая критическое поведение комплексной электропроводности в зависимости от частоты и доли металлического объема, выражается через два индекса, известных из расчетов и модельных экспериментов. Проанализированы результаты численных расчетов Вебмана и др.

1. Introduction

It is known that near the metal–non-metal transition (MNMT) many systems consist of randomly distributed metallic and dielectric regions. The volume fraction of metallic regions increases with some physical parameter (composition, temperature, light intensity) and MNMT occurs when this fraction corresponds to the percolation threshold for the metallic regions [1].

Most completely the conductivity of such a system was studied in the case of zero conductivity of dielectric regions. For example it is a well-known problem of the effective conductivity of a wire lattice with randomly removed bonds between nearest sites (bond problem). Let the fraction of removed bonds be $1 - x$ and the percolation threshold be x_c. It is clear that the effective conductivity $\sigma(x) = 0$ if $x < x_c$. If $x > x_c$ the conductivity increases with x as

$$\sigma(x) = \sigma_M(x - x_c)^t\,, \tag{1}$$

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where σ_M is the conductivity of the lattice with all bonds present ($x = 1$). (We omit the numerical coefficients which have to be in (1) and in the following equations.)

It is usually supposed that the critical index t does not depend on the type of the percolation problem, but it depends on the dimensionality of space [1, 2] (universality hypothesis). This means for example that the index t is the same for the plane wire lattice and for the graphite paper with holes randomly punched in it [3]. The results of computer and model experiments do not contradict the universality hypothesis. These results give for the three-dimensional case $t_3 = 1.6$ and for the two-dimensional case $t_2 = 1.3$. The connection between the index t and the correlation radius index is discussed in papers [2, 4].

The first problem considered in this paper is the generalization of (1) for the case of non-zero conductivity of dielectric regions. In terms of the wire lattice model this generalization means that the randomly chosen metallic bonds between nearest neighbours are not removed, but they are replaced by bonds with smaller conductances. Let the conductivity of the lattice with all metallic bonds replaced be σ_D and let us assume that $h \equiv \sigma_D/\sigma_M \ll 1$. It is clear that $\sigma(x)$ is a regular function of x for any small but non-zero value of h. Thus the parameter h plays the same role as the magnetic field in the ferromagnetic phase transition theory. The first question is what is the order of magnitude of $\sigma(x)$ at the point $x = x_c$. In close analogy with the phase transition theory we suppose that $\sigma(x_c)$ obeys a power law

$$\sigma(x_c) = \sigma_M \left(\frac{\sigma_D}{\sigma_M}\right)^s = \sigma_M h^s . \tag{2}$$

So we introduce a new critical index s. In the two-dimensional case one can determine this index from an exact result obtained by Dykhne [5]. Dykhne has found that in a two-component system with symmetrical distribution of both components (metal and dielectric) $\sigma(x_c) = (\sigma_D\sigma_M)^{1/2}$, i.e. $s = 1/2$. Following the universality hypothesis we suppose that $s = 1/2$ for all two-dimensional percolation problems. There are no exact results for the three-dimensional case. The computer calculations performed by Webman, Jortner, and Cohen [6] (W.J.C.) show that in this case also $\sigma(x_c) \gg \sigma_D$, i.e. $s < 1$. The effective medium approximation gives $s = 1/2$ independently of space dimensionality but it is known [1] that this approximation does not work near the percolation threshold.

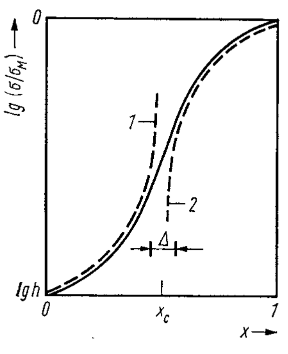

Fig. 1. The theoretical dependence of $\sigma(x)$ for the (σ_M, σ_D) problem (solid line). (1) Equation (1); (2) equation (3)

If $\sigma(x_c) \gg \sigma_D$ the conductivity $\sigma(x)$ has to increase with x for $x < x_c$, too (Fig. 1). Following W.J.C. we write $\sigma(x)$ for $x < x_c$ in the form

$$\sigma(x) = \sigma_D(x_c - x)^{-q}; \quad q > 0 , \tag{3}$$

where q is another critical index. Equation (3) has to be valid if $\sigma(x) \ll \sigma(x_c)$. A smooth transition from (1) to (3) occurs in some small interval Δ near the point x_c (Fig. 1). It means that (3) is valid if $x_c - x \gg \Delta$. Equation (1) works if $\sigma(x_c) \ll \ll \sigma(x) \ll \sigma_M$ or if $x - x_c \gg \Delta$. The result of W.J.C. calculations is that $q_3 \approx 1$. (The effective medium approximation gives $q = t = 1$ independently of space dimensionality.)

In the next section we put forward a scaling hypothesis for the function $\sigma(x, h)$. This hypothesis gives the following relation between indexes t, s, and q:

$$q = t\left(\frac{1}{s} - 1\right). \tag{4}$$

In the two-dimensional case $s_2 = 1/2$ and it follows from (4) that $q_2 = t_2$. In the three-dimensional case $q_3 \approx 1$, $t_3 \approx 1.6$ and we obtain from (4) that $s_3 \approx 0.62$. We show that this result is in agreement with the W.J.C. computer experiment. We discuss also the role of the finite size effects in this experiment.

Up to this point we discussed the problem of the effective dc conductivity of a two-component system. This we call for brevity (σ_M, σ_D) problem. In Section 3 we consider the properties of the complex ac conductivity near the percolation threshold. The problem is to find the effective conductivity of a two-component system

$$\sigma(\omega, x) = \mathrm{Re}\, \sigma(\omega, x) - \frac{i\omega}{4\pi}\, \varepsilon(\omega, x) , \tag{5}$$

where $\varepsilon(\omega, x)$ is the real dielectric constant and ω is the frequency. We consider the low-frequency case, and up to Section 5 we suppose that the conductivity of the metallic component σ_M is real. In Section 5 we discuss the influence of the imaginary part of σ_M. The conductivity of the dielectric component has the form

$$\sigma_D(\omega) = \frac{\omega}{4\pi i}\, \varepsilon_0 + \sigma_D^0 . \tag{6}$$

We assume that the dc conductivity σ_D^0 is much smaller than σ_M but we suppose nothing about the relation between σ_D and $\omega\varepsilon_0/4\pi$. This we call $[\sigma_M, \sigma_D(\omega)]$ problem.

Now we discuss the results of our paper concerning the simplest case, $\sigma_D^0 = 0$. If $\omega = 0$ the conductivity $\sigma(x)$ obeys the law (1) if $x > x_c$ and equals zero if $x < x_c$. However, $\sigma(\omega, x)$ is non-zero for any value of x if $\omega \neq 0$. In this sense the parameter ω/σ_M is similar to the parameter $h = \sigma_D/\sigma_M$ in the (σ_M, σ_D) problem. The total description of the function $\mathrm{Re}\, \sigma(\omega, x)$ near the percolation threshold x_c includes three critical indexes s, p, and t. Index s characterizes the frequency dependence $\mathrm{Re}\, \sigma(\omega, x)$ at the point $x = x_c$

$$\mathrm{Re}\, \sigma(\omega, x_c) = \sigma_M \left(\frac{\omega\varepsilon_0}{4\pi\sigma_M}\right)^s . \tag{7}$$

The dependence upon $x - x_c$ has the form

$$\mathrm{Re}\, \sigma(\omega, x) = \frac{\omega^2\varepsilon_0^2}{4\pi\sigma_M(x_c - x)^p} \tag{8}$$

if $x < x_c$ and $\mathrm{Re}\,\sigma(\omega, x) \ll \mathrm{Re}\,\sigma(\omega, x_c)$. The last expression describes the dissipation of energy due to the polarization of isolated metallic clusters. That is why $\mathrm{Re}\,\sigma \sim \omega^2$. Finally $\mathrm{Re}\,\sigma$ is given by (1) if $x > x_c$ and $\mathrm{Re}\,\sigma(\omega, x) \gg \mathrm{Re}\,\sigma(\omega, x_c)$. A smooth transition from (8) and (1) to (7) takes place in a small interval near the point x_c.

We show in Section 3 that the $[\sigma_M, \sigma_D(\omega)]$ problem can be reduced to the (σ_M, σ_D) problem. It follows from this reduction that the index s in (7) is just the same as in (2) and that the index p is related with s and t by

$$p = t\left(\frac{2}{s} - 1\right). \tag{9}$$

The behaviour of the dielectric constant ε near the threshold is rather extraordinary. Mott and Davis [7] put forward microscopic arguments for the divergence of the dielectric constant near the MNMT threshold. There is some experimental evidence of such a divergence.

A sharp increase of the dielectric constant near the MNMT threshold was observed in doped Si [8] and in VO_2 films [9]. The anomaly in the dielectric properties of disordered systems was discussed also by Bonch-Bruevich [10].

Dubrov et al. [11] considered a two-component system with a percolation MNMT. They used the effective medium approximation and showed that the static dielectric constant near the threshold has the form

$$\varepsilon(0, x) = \frac{\varepsilon_0}{|x - x_c|^q}. \tag{10}$$

In the framework of the effective medium approximation $q = 1$ independently of space dimensionality. Dubrov et al. [11] performed a very interesting model experiment which demonstrated the increase of the dielectric constant near the percolation threshold in the two-dimensional case. They gave also a qualitative interpretation of this phenomenon which is as follows. Near the percolation threshold the metallic clusters are separated by thin dielectric regions. Each pair of nearest clusters forms a condenser whose effective surface tends to infinity near the percolation threshold. Then the effective capacity of the system diverges, too.

In Section 3 we show that the index q in (10) is just the same as in (3) and in (4). In the two-dimensional case $q_2 = t_2$. Then $\varepsilon \sim |x - x_c|^{-t_2}$. In the three-dimensional case $\varepsilon \sim |x - x_c|^{-q_3}$ and $q_3 \approx 1$.

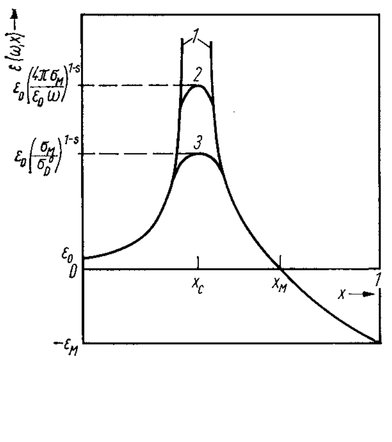

Fig. 2. Dielectric constant $\varepsilon(\omega, x)$. (1) $\omega = 0$, $\sigma_D^0 = 0$ (equation (10)). Curves 2 and 3 correspond to the same frequency ω. (2) $4\pi\sigma_D^0/\varepsilon_0 \ll \omega$, (3) $4\pi\sigma_D^0/\varepsilon_0 \gg \omega$. The behaviour of curves for $x > x_M$ is explained in Section 5

The dielectric constant is finite for any small but non-zero value of frequency ω. We show that

$$\varepsilon(\omega, x_c) = \varepsilon_0 \left(\frac{4\pi\sigma_M}{\varepsilon_0\omega}\right)^{1-s}. \tag{11}$$

Equation (10) is valid for $\varepsilon(\omega, x)$ if $\varepsilon(\omega, x) \ll \varepsilon(\omega, x_c)$. However, (10) fails in a small interval of x near the point x_c where $\varepsilon(\omega, x)$ tends to the finite value (11) (Fig. 2).

So we show that the dielectric constant diverges near the percolation threshold and we find the critical indexes without the effective medium approximation.

It is important that the indexes t, s, and q depend on the dimensionality of space, but the relation (4) is universal. Moreover, this relation can be established more generally without reduction to the (σ_M, σ_D) problem. One can suppose that the complex conductivity is a regular function of x for any non-zero value of frequency ω and that the characteristic interval Δ is the same for the real and for the imaginary parts of the conductivity. Then Kramers-Kronig relations lead to (4) and (9). This way does not permit to find the numerical values of the indexes which are known from the (σ_M, σ_D) problem only. However, this means that (4) and (9) describe a more general property MNMT concerning not only two-component systems with percolation threshold.

In Section 4 we discuss the influence of the non-zero value of σ_D^0. Most interesting is that an anomaly of the dielectric constant near the threshold takes place even if $\omega \ll 4\pi\sigma_D^0/\varepsilon_0$. The inequality $\sigma_D^0 \ll \sigma_M$ is only necessary.

2. (σ_M, σ_D) Problem

The effective conductivity σ of a two-component system (σ_M, σ_D) divided by σ_M depends on two variables which are $h = \sigma_D/\sigma_M$ and $\tau = x - x_c$ (x is the volume fraction of the metallic component). The conductivity is a regular function of τ for any non-zero value of h. We suppose that the broadening of the singularity which takes place if $h = 0$ is described by a single parameter Δ. Equation (1) is valid if $\tau \gg \Delta$ and a deviation from (1) becomes important if $\tau \approx \Delta$. This Δ must be a positive power of h, i.e. $\Delta = h^m$. Then we write the conductivity in the usual scaling form

$$\bar{\sigma} \equiv \frac{\sigma}{\sigma_M} = h^s \varphi\left(\frac{\tau}{h^m}\right), \tag{12}$$

where φ is an unknown function. The conductivity is non-zero at the point $\tau = 0$. Then we can put $\varphi(0) = 1$. (We omit all numerical coefficients.) Equation (12) is in close analogy with scaling laws of the phase transition theory. The magnetization depends on magnetic field h and temperature $(T - T_c)/T_c$ in the same way.

It was mentioned that $\bar{\sigma} = \tau^t$ if $\tau > 0$ and $\tau/h^m \to \infty$. Then

$$\varphi(z) \underset{z\to\infty}{=} z^t \quad \text{and} \quad m = \frac{s}{t}. \tag{13}$$

Now we can find the form of the function $\varphi(z)$ for large negative values of z. For small x we have $\sigma = \sigma_D$, i.e. $\bar{\sigma} = h$. It follows then

$$\varphi(z) \underset{z \to -\infty}{=} \frac{1}{(-z)^q}\,; \qquad q = \frac{t}{s} - t \tag{14}$$

and we obtain (3) which is valid if $\tau < 0$ and $|\tau| \gg h^{s/t}$. Finally (2) follows from (12) if $\tau = 0$.

Now we can compare our results with the W.J.C. computer experiment. The conductivity of an $18 \times 18 \times 18$ cube was computed. Every lattice bond has unity conductance with probability x (metallic bonds) and has a conductance $h \ll 1$ with probability $1 - x$ (dielectric bonds). Typical results of calculation are reproduced on Fig. 3 for $h = 1.2 \times 10^{-3}$ and $h = 10^{-5}$. The spatial distribution of metallic and dielectric bonds was correlated. This correlation itself does not prevent from applying our theory. Unfortunately the exact value of the percolation threshold x_c becomes unknown due to this correlation. W.J.C. defined it from their data. For the case reproduced in Fig. 3 $x_c = 0.18$.

First of all we see a discontinuous drop on curve 2 (Fig. 3) in contrast with the theoretical curve (Fig. 1). The discontinuity was observed earlier in model experiments [12]. Our understanding is that this phenomenon is connected with the finite size of the system. It is known [2] that near the percolation threshold the metallic network becomes extremely rare. Then if τ is small enough only one metallic channel remains between the opposite boundaries of the finite-size sample. But the metallic network determines the conductivity if $\tau > \Delta$. Then the break of the last channel gives a discontinuous drop of the observable conductivity. We can estimate the values $\bar{\sigma}_1$ and $\bar{\sigma}_2$ which characterize the drop of curve 2 (Fig. 3). It is known [2] that the characteristic size of the random network consisting of connected metallic bonds is of the order

$$L(\tau) = \frac{1}{|\tau|^\nu} \tag{15}$$

(in units of the lattice constant). This is the correlation radius of the percolation theory [2]. Index $\nu \approx 0.9$ in the three-dimensional case. It is clear that the finite size of a "sample" becomes important when the correlation radius $L(\tau)$ reaches the size of the cube l. The corresponding value of τ is

$$|\tau| \approx l^{-1/\nu} \equiv W_l\,. \tag{16}$$

The number of metallic channels inside the cube is of the order of unity for this τ. So we can estimate the conductivity $\bar{\sigma}_1$ which precedes the break of the last

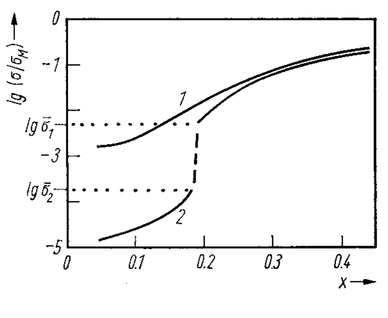

Fig. 3. The results of the W.J.C. [6] computer experiment. (1) $h = 1.2 \times 10^{-3}$; (2) $h = 10^{-5}$

channel. Insertion of (16) into (1) gives

$$\bar{\sigma}_1 = \frac{\sigma_1}{\sigma_M} = l^{-t/\nu} . \tag{17}$$

Taking $t_3 = 1.6$, $\nu_3 = 0.9$ for $l = 18$ we obtain $\lg \bar{\sigma}_1 = -2.3$ which agrees exactly with the W.J.C. result.

It is important that the percolation threshold x_{cl} of a cube with size l is an uncertain value which can fluctuate from one set of random variables to another. Its dispersion is of the order of W_l (16). Equations (1) and (3) which describe infinite systems are valid if $|\tau| \gg W_l$. Then one can estimate the conductivity $\bar{\sigma}_2$ which follows the break of the last metallic channel by insertion of (16) into (3). This gives

$$\bar{\sigma}_2 = \frac{\sigma_2}{\sigma_M} = h l^{q/\nu} . \tag{18}$$

Taking $q_3 = 1$, $\nu_3 = 0.9$ for $l = 18$ and $h = 10^{-5}$ we obtain $\lg \bar{\sigma}_2 = -3.6$. The W.J.C. result is $\lg \bar{\sigma}_2 = -3.7$.

Now we explain the absence of discontinuity on curve 1 of Fig. 3 which corresponds to a larger value of h. It was emphasized that (1) and (3) are valid only if $|\tau|$ is much larger than the interval $\Delta = h^{s/t}$ (see Fig. 1). If $|\tau| \ll \Delta$ the values of conductivity corresponding to τ and $-\tau$ are nearly equal and are given by (2). The discontinuity is absent if W_l is smaller than Δ. One can estimate the maximum value of h for which a discontinuity takes place by solving the equation $W_l = h^{s/t}$. This gives

$$h_l = l^{-t/s\nu} . \tag{19}$$

Taking $t_3 = 1.6$, $s_3 = 0.62$, $\nu_3 = 0.9$ for $l = 18$ we obtain $\lg h_{18} = -3.6$. That is why a discontinuity indeed has to be absent if $h = 1.2 \times 10^3$, and one can conclude that curve 1 corresponds to an infinite lattice. Then we can determine the index s from this curve. Taking $x_c = 0.18$ we find that $\lg \sigma(x_c) = -1.85$. Insertion of $h = 1.2 \times 10^{-3}$ into (2) gives $s_3 = 0.65$ which agrees with the above result $s_3 = 0.62$.

3. $[\sigma_M, \sigma_D(\omega)]$ Problem

Now we suppose that the conductivity of the dielectric component is given by (6). One can reduce this new problem to the previous one considering complex values of the frequency ω. The conductivity $\sigma_D(\omega)$ is real and positive if $\operatorname{Re} \omega = 0$, $\operatorname{Im} \omega > 0$. (Only the case $\varepsilon_0 > 0$ is considered.) For these values of ω we obtain using (2)

$$\bar{\sigma}(\omega, x_c) = \frac{\sigma(\omega, x_c)}{\sigma_M} = \left(\frac{\sigma_D(\omega)}{\sigma_M}\right)^s . \tag{20}$$

It is well-known that imaginary values of ω with $\operatorname{Im} \omega > 0$ describe an exponentially increasing current and that capacitances for this current are equivalent with resistances for the direct current. So (20) has a simple physical interpretation. But $\bar{\sigma}(\omega)$ has to be an analytic function within the upper half plane of ω. Then we can perform an analytic continuation which shows that (20) is valid for real values of ω, too. Now we can find the real and imaginary parts of (20). We begin with the case $\sigma_D^0 = 0$, $\sigma_D(\omega) = -i\omega\varepsilon_0/4\pi$. Then using the definition (5) we obtain (7) and (11).

Near the percolation threshold the complex conductivity $\bar{\sigma}(\omega, x)$ depends on two variables. One is the complex variable $h(\omega) = \sigma_D(\omega)/\sigma_M$ and the other is the real one $\tau = x - x_c$. Using the same scaling arguments as we used in the (σ_M, σ_D) problem we write $\bar{\sigma}$ in the form

$$\bar{\sigma} = h^s(\omega)\, \psi\left(\frac{\tau}{h^m}\right), \tag{21}$$

where ψ is an unknown function and $\psi(0) = 1$. The function $\psi(z)$ is equal to $\varphi(z)$ (12) for real z. It is real if $\operatorname{Re}\omega = 0$, $\operatorname{Im}\omega > 0$. Then it follows from (12) and (13) that $m = s/t$. The function (21) can be obtained within the upper half plane of ω as analytic continuation of the function (12) from the imaginary axis. We suppose for simplicity that $\sigma_D^0 = 0$ and we return to the general case in Section 4. Then

$$z = \frac{\tau}{h^{s/t}} = \tau \left[\frac{-i\omega\varepsilon_0}{4\pi\sigma_M}\right]^{-s/t}. \tag{22}$$

Let us find the form of $\psi(z)$ for large z, i.e. for $\omega \to 0$. If $\omega = |\omega| \exp(i\pi/2)$ and $\tau < 0$ we obtain $\arg z = \pi$. Then $\psi(z) = \varphi(z)$ and for large $|z|$ it is determined by (14). But $h^s\psi(z)$ is an analytic function of ω in the upper half plane. Then we obtain

$$\psi(z) = z^{-q}; \qquad q = \frac{t}{s} - t \tag{23}$$

for $|z| \gg 1$ and for all $\arg z$ in the interval

$$\pi - \frac{s}{t}\frac{\pi}{2} \leqq \arg z \leqq \pi + \frac{s}{t}\frac{\pi}{2} \tag{24}$$

which corresponds to $0 < \arg \omega < \pi$. Then we obtain for real ω and $\tau < 0$ $\left(\arg z = \pi + \frac{s}{t}\frac{\pi}{2}\right)$

$$\bar{\sigma} = -\frac{i\omega\varepsilon_0}{4\pi\sigma_M} |\tau|^{-q}; \qquad \tau < 0\,.$$

Using the definition (5) we obtain

$$\varepsilon(\omega, x) = \frac{\varepsilon_0}{(-\tau)^q}; \qquad \tau < 0 \tag{25}$$

which agrees with (10). It is valid if $|\tau| \gg (\omega\varepsilon_0/4\pi\sigma_M)^{s/t}$.

In the case $\tau > 0$ we have $\arg z = 0$ if $\arg \omega = \pi/2$. Then $\psi(z) = \varphi(z)$ and for large z it is determined by (13). This gives

$$\psi(z) = z^t \quad \text{for } |z| \gg 1 \tag{26}$$

and

$$-\frac{s}{t}\frac{\pi}{2} \leqq \arg z \leqq \frac{s}{t}\frac{\pi}{2}. \tag{27}$$

Equation (26) leads to (1).

The expressions (23) and (26) for $\psi(z)$ are different because $q \neq -t$. Then the intervals (24) and (27) cannot include common points. This condition is fulfilled if $s/t < 1$. So we obtain a new inequality relating indexes s and t. The values of indexes discussed above satisfy this inequality.

In the lowest approximation (23) and (26) the conductivity $\bar{\sigma}$ is real for $\tau > 0$ and is imaginary for $\tau < 0$. Now we find the next terms of the expansion of $\psi(z)$ for large $|z|$. If $\tau < 0$ the next term describes the dissipation of energy inside the isolated metallic clusters which are immersed in the dielectric medium. It is clear that it has to be $\operatorname{Re}\sigma \sim \omega^2$. Equation (23) gives $\sigma \sim i\omega$. Then the ratio of the next term to the main term has to be proportional to $i\omega$, i.e. $z^{-t/s}$. This gives

$$\psi(z) = \frac{1}{z^q}[1 + \text{const } z^{-t/s}]\,. \tag{28}$$

This equation is valid if $|z| \gg 1$ within the interval (24). Then we substitute (22) into (28) and (21) and obtain the result (8). Now we find the correction term in (26) which determines the dielectric constant in the case $\tau > 0$. The dielectric constant $\varepsilon(\omega, \tau)$ must tend to a non-zero and finite limit when $\omega \to 0$ and τ is non-zero. This means that $\operatorname{Im}\sigma \sim \omega$. Equation (26) leads to real σ which does not depend on ω. Then again the ratio of the next term to the main term is proportional to $i\omega$

$$\psi(z) = z^t\,(1 + \text{const } z^{-t/s})\,. \tag{29}$$

Equations (29), (22), and (21) lead to (25) in the case $\tau > 0$, $\tau \gg (\omega\varepsilon_0/4\pi\sigma_M)^{s/t}$. So we have shown that the index q in (25) is the same for $\tau > 0$ and $\tau < 0$.

4. Non-Zero DC Conductivity of a Dielectric

In the preceding section we assumed $\sigma_D^0 = 0$. Now we discuss the most interesting consequences of the non-zero value of σ_D^0. If $\tau = 0$ $(x = x_c)$ one can use (20) and (6). This gives

$$\bar{\sigma}(x_c) = \frac{\left(\sigma_D^0 - \dfrac{i\omega\varepsilon_0}{4\pi}\right)^s}{\sigma_M}\,. \tag{30}$$

One can see that our results (7) and (11) are valid if $4\pi\sigma_D^0/\varepsilon_0 \ll \omega$. In the opposite case $4\pi\sigma_D^0/\varepsilon_0 \gg \omega$ the real conductivity $\operatorname{Re}\sigma(x_c)$ is given by (2) and does not depend on frequency. The unexpected result is that in this case $\varepsilon(\omega, x_c)$ is large (curve 3 in Fig. 2). It follows from (30)

$$\varepsilon(\omega, x_c) = \varepsilon_0\left(\frac{\sigma_M}{\sigma_D^0}\right)^{1-s}; \qquad \sigma_D^0 \gg \frac{\omega\varepsilon_0}{4\pi}\,. \tag{31}$$

Thus a sharp increase of $\varepsilon(\omega, x)$ near the MNMT threshold can be observed at any small frequencies if only $\sigma_D^0 \ll \sigma_M$. One can obtain $\bar{\sigma}(\omega, \tau)$ substituting $h = \sigma_D(\omega)/\sigma_M$ in (21) and using (28) for $\tau > 0$ and (29) for $\tau < 0$. In general the broadening of the singularity of $\sigma(\omega, \tau)$ near the percolation threshold occurs in the τ interval of the order of $\Delta = |(\sigma_D^0/\sigma_M) - (i\omega\varepsilon_0/4\pi\sigma_M)|^{s/t}$. Thus the interval is determined by the largest from these two terms (see Fig. 2).

The behaviour of $\operatorname{Re}\sigma$ at low frequencies in the case $\tau < 0$ is also interesting. One can obtain from (28) and (21) that $\operatorname{Re}\sigma$ is the sum of two terms. The first term is given by (3) with $\sigma_D = \sigma_D^0$ and the second one is given by (8). The first term is the main one at sufficiently small frequencies.

5. The Role of Im σ_M

It was supposed above that the conductivity of the metal, σ_M, is real. However, free electrons inevitably create an imaginary part so that $\sigma_M(\omega) = \sigma_M^0 - (\varepsilon_M\omega/4\pi i)$. The Drude model for example gives $\varepsilon_M = \sigma_M^0\tau_p$ where τ_p is the relaxation time. For sufficiently small frequencies Im $\sigma_M \ll \sigma_M^0$. One can show that a non-zero Im σ_M does not change the above results if $x \leq x_c$. To do this one can replace σ_M by $\sigma_M(\omega)$ in (20), (23), and (28). If, however, $\varepsilon_M \gg \varepsilon_0$ (25) for the dielectric constant is changed for $x > x_c$ sufficiently far from the transition point x_c. Indeed $\varepsilon(\omega, x)$ has to tend to $-\varepsilon_M$ if $x \to 1$ rather than to a value of the order of ε_0 which follows from (25). Replacing σ_M by $\sigma_M(\omega)$ in (21), (22), and (29) we obtain

$$\varepsilon(\omega, \tau) = \varepsilon_0\tau^{-q} - \varepsilon_M\tau^t\,. \tag{32}$$

The second term in (32) becomes important far from the transition point if $x \gtreqqless x_M = x_c + (\varepsilon_0/\varepsilon_M)^{1/q+t}$. This term leads to negative $\varepsilon(\omega,\tau)$ for large τ $\left(\varepsilon(\omega,\tau) \approx -\varepsilon_M \text{ if } \tau \approx 1\right)$. Of course (32) is valid only if $\tau \ll 1$.

6. Specific Features of Two-Dimensional Problems

When we discuss the conductivities $\sigma(\omega, x)$, σ_M, and σ_D in the two-dimensional case we mean the two-dimensional conductivities which are the conductances of squares with unit area. These conductivities are measured in units of Ω^{-1}. Then we see from (5) and (6) that $\varepsilon(\omega, x)$ and ε_0 are not dimensionless and these values need special discussion.

Let us consider a two-component film where the local conductivity does not depend on the coordinate perpendicular to the film. This is just the two-dimensional system for the percolation theory. The two-dimensional film conductivity σ_2 is obtained from the bulk conductivity σ_3 using the relation $\sigma_2 = \sigma_3 d$ where d is the film thickness. The same relation can be used also for the dielectric constant $\varepsilon_2 = \varepsilon_3 d$. But it is clearly valid only if the total current (with the displacement current) flows mainly inside the film. This condition is fulfilled if the film thickness is large enough or if the conductivity $\sigma_D(\omega)$ is large. The current leaves the film and flows in vacuum around the dielectric regions as displacement current in the opposite case of an extremely thin film (Fig. 4). The relation $\varepsilon_2 = \varepsilon_3 d$ is meaningless in this case. Moreover, the $[\sigma_M, \sigma_D(\omega)]$ problem cannot be reduced to the two-dimensional (σ_M, σ_D) problem in such a case. But it can be reduced to another interesting dc problem. Let us consider a metallic film with holes randomly punched in it (or a two-dimensional wire lattice with partly removed bonds). We obtain the two-dimensional (σ_M, σ_D) problem if this film (or lattice) is pressed to conductive paper. However, we obtain a new $(\sigma_{M_2}, \sigma_{D_3})$ problem if the film is in a three-dimensional medium with non-zero conductivity. For example it can be immersed in a conductive liquid.[2])

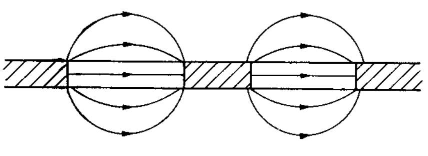

Fig. 4. The distribution of the electric field in a two-component thin film. The metallic regions are shaded

[2]) This technical idea belongs to M. E. Levinshtein.

The $[\sigma_M, \sigma_D(\omega)]$ problem for thin films is obviously reduced to the $(\sigma_{M_2}, \sigma_{D_3})$ problem.

We can formulate the scaling law for the $(\sigma_{M_2}, \sigma_{D_3})$ problem and show that the indexes are connected by (4). However, the numerical values of the indexes for this problem are unknown.

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